

Relativistic r -modes in Slowly Rotating Neutron Stars: Numerical Analysis in the Cowling Approximation

Shijun Yoshida¹ and Umin Lee

*Astronomical Institute, Graduate School of Science, Tohoku University, Sendai 980-8578,
Japan*

yoshida@astr.tohoku.ac.jp, lee@astr.tohoku.ac.jp

ABSTRACT

We investigate the properties of relativistic r -modes of slowly rotating neutron stars by using a relativistic version of the Cowling approximation. In our formalism, we take into account the influence of the Coriolis like force on the stellar oscillations, but ignore the effects of the centrifugal like force. For three neutron star models, we calculated the fundamental r -modes with $l' = m = 2$ and 3. We found that the oscillation frequency $\bar{\sigma}$ of the fundamental r -mode is in a good approximation given by $\bar{\sigma} \approx \kappa_0 \Omega$, where $\bar{\sigma}$ is defined in the corotating frame at the spatial infinity, and Ω is the angular frequency of rotation of the star. The proportional coefficient κ_0 is only weakly dependent on Ω , but it strongly depends on the relativistic parameter GM/c^2R , where M and R are the mass and the radius of the star. All the fundamental r -modes with $l' = m$ computed in this study are discrete modes with distinct regular eigenfunctions, and they all fall in the continuous part of the frequency spectrum associated with Kojima's equation (Kojima 1998). These relativistic r -modes are obtained by including the effects of rotation higher than the first order of Ω so that the buoyant force plays a role, the situation of which is quite similar to that for the Newtonian r -modes.

Subject headings: instabilities — stars: neutron — stars: oscillations — stars: rotation

1. Introduction

It is Andersson (1998) and Friedman & Morsink (1998) who realized that the r -modes in rotating stars are unstable against the gravitational radiation reaction. Since then, a large

¹Research Fellow of the Japan Society for the Promotion of Science.

number of papers have been published to explore the possible importance of the instability in neutron stars. The r -modes in rotating stars are restored by the Coriolis force. They have the dominant toroidal component of the displacement vector and their oscillation frequencies are comparable to the rotation frequency Ω of the star (see, e.g., Bryan 1889; Papaloizou & Pringle 1978; Provost, Berthomieu, & Roca 1981; Saio 1982; Unno et al. 1989). For the r -mode instability, see reviews by, e.g., Friedman & Lockitch (1999), Andersson & Kokkotas (2001), Lindblom (2001), and Friedman & Lockitch (2001). Most of the studies on the r -mode instability in neutron stars, however, have been done within the framework of Newtonian dynamics. Since the relativistic factor can be as large as $GM/c^2R \sim 0.2$ for neutron stars where M and R are respectively the mass and the radius, the relativistic effects on the r -modes are essential.

From time to time the effects of general relativity on the r -modes in neutron stars have been discussed in the context of the r -mode instability. Kojima (1998) is the first who investigated the r -modes in neutron stars within the framework of general relativity. In the slow rotation approximation, Kojima (1998) derived a second order ordinary differential equation governing the relativistic r -modes, expanding the linearized Einstein equation to the first order of Ω , and assuming that the toroidal component of the displacement vector is dominant and the oscillation frequency is comparable to Ω . Kojima (1998) showed that this equation has a singular property and allows a continuous part in the frequency spectrum of the r -modes (see, also, Beyer & Kokkotas 1998). Recently, Lockitch, Andersson, & Friedman (2001) showed that Kojima’s equation is appropriate for non-barotropic stars but not for barotropic ones. They found that both discrete regular r -modes and continuous singular r -modes are allowed in Kojima’s equation for uniform density stars, and suggested that the discrete regular r -modes are a relativistic counterpart of the Newtonian r -modes. Yoshida (2001) and Ruoff & Kokkotas (2001a) showed that discrete regular r -mode solutions to Kojima’s equation exist only for some restricted ranges of the polytropic index and the relativistic factor for polytropic models, and that regular r -mode solutions do not exist for the typical ranges of the parameters appropriate for neutron stars.

The appearance of continuous singular r -mode solutions in Kojima’s equation (Kojima 1998) caused a stir in the community of people who are interested in the r -mode instability. Lockitch et al. (2001) (see also Beyer & Kokkotas 1999) suggested that the singular property in Kojima’s equation could be avoided if the energy dissipation associated with the gravitational radiation is properly included in the eigenvalue problem because the eigenfrequencies become complex due to the dissipations and the singular point can be detoured when integrated along the real axis. Very recently, this possibility of avoiding the singular property in Kojima’s equation has been examined by Yoshida & Futamase (2001) and Ruoff & Kokkotas (2001b), who showed that the basic properties of Kojima’s equation do not change even if the

gravitational radiation reaction effects are approximately included into the equation. Quite interestingly, as shown by Kojima & Hosonuma (2000), if the third order rotational effects are added to the original Kojima’s equation (Kojima 1998), the equation for the r -modes becomes a fourth order ordinary linear differential equation, which has no singular properties if the Schwarzschild discriminant associated with the buoyant force does not vanish inside the star. By solving a simplified version of the extended Kojima’s equation for a simple toy model, Lockitch & Andersson (2001) showed that because of the higher order rotational terms the singular solution in the original Kojima’s equation can be avoided. Obviously, we need to solve the complete version of the extended Kojima’s equation to obtain a definite conclusion concerning the continuous singular r -mode solutions.

It may be instructive to turn our attention to a difference in mathematical property between the Newtonian r -modes and the relativistic r -modes associated with Kojima’s equation. It is usually assumed that in the lowest order of Ω the eigenfunction as well as the eigenfrequency of the r -modes is proportional to Ω . In the case of the Newtonian r -modes, if we employ a perturbative method for r -modes in which the angular frequency Ω is regarded as a small expanding parameter, the radial eigenfunctions of order of Ω can be determined by solving a differential equation derived from the terms of order of Ω^3 , which bring about the couplings between the oscillations and the buoyant force in the interior (e.g., Provost et al 1981, Saio 1982). In other words, there is no differential equation, of order of Ω , which determines the radial eigenfunction of the Newtonian r -modes. On the other hand, in the case of the general relativistic r -modes derived from Kojima’s equation, we do not have to take account of the rotational effects of order of Ω^3 to obtain the radial eigenfunctions of order of Ω . That is, the eigenfrequency and eigenfunction of the r -modes are both determined by a differential equation (i.e., Kojima’s equation) derived from the terms of order of Ω . This remains true even if we take the Newtonian limit of Kojima’s equation to calculate the r -modes (Lockitch et al. 2001; Yoshida 2001). We think that this is an essential difference between the Newtonian r -modes and the relativistic r -modes associated with Kojima’s equation. Considering that Kojima’s equation can give no relativistic counterpart of the Newtonian r -mode, it is tempting to assume that some terms representing certain physical processes are missing in the original Kojima’s equation. On the analogy of the r -modes in Newtonian dynamics, we think that the buoyant force in the interior plays an essential role to obtain a relativistic counterpart of the Newtonian r -modes and that the terms due to the buoyant force will appear when the rotational effects higher than the first order of Ω are included. This is consistent with the suggestions made by Kojima & Hosonuma (2000) and Lockitch & Andersson (2001). In this paper, we calculate relativistic r -modes by taking account of the effects of the buoyant force in a relativistic version of the Cowling approximation, in which all the metric perturbations are omitted. In our formulation, all the terms

associated with the Coriolis force are included but the terms from the centrifugal force are all ignored. Our formulation can take account of the rotational contributions, due to the Coriolis force, higher than the first order of Ω . Note that our method of solution is not a perturbation theory in which Ω is regarded as a small expanding parameter for the eigenfrequency and eigenfunction. Similar treatment has been employed in Newtonian stellar pulsations in rotating stars (see, e.g., Lee & Saio 1986; Unno et al. 1989; Bildsten, Ushomirsky & Cutler 1996; Yoshida & Lee 2001). This treatment is justified for low frequency modes because the Coriolis force dominates the centrifugal force in the equations of motion. The plan of this paper is as follows. In §2, we describe the formulation of relativistic stellar pulsations in the relativistic Cowling approximation. In §3, we show the modal properties of the r -modes in neutron star models. In §4, we discuss about our results, specially the effect of the buoyancy in the fluid core on the modes. §5 is devoted to conclusions. In this paper, we use units in which $c = G = 1$, where c and G denote the velocity of light and the gravitational constant, respectively.

2. Formulation

2.1. Equilibrium State

We consider slowly and uniformly rotating relativistic stars in equilibrium. If we take account of the rotational effects up to first order of Ω , the geometry in the stars can be described by the following line element (see, e.g. Thorne 1971):

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 - 2\omega(r) r^2 \sin^2 \theta dt d\varphi. \quad (1)$$

The fluid four-velocity in a rotating star is given by

$$u^\alpha = \gamma(r, \theta) (t^\alpha + \Omega \varphi^\alpha), \quad (2)$$

where t^α and φ^α stand for the timelike and rotational Killing vectors, respectively. Here the function γ is chosen to satisfy the normalization condition $u^\alpha u_\alpha = -1$. If we consider the accuracy up to order of Ω , the function γ reduces to:

$$\gamma = e^{-\nu(r)}. \quad (3)$$

Once the physical quantities of the star such as the pressure, $p(r)$, the mass-energy density, $\rho(r)$, and the metric function, $\nu(r)$, are given, the rotational effect on the metric, $\omega(r)$ can be obtained from a well-known numerical procedure (see, e.g., Thorne 1971).

2.2. Pulsation Equations in the Cowling Approximation

General relativistic pulsation equations are usually obtained by linearizing Einstein's field equation. The linearized Einstein equation contains perturbations associated with the metric fluctuations and the fluid motions. In this paper, to simplify the problem, we employ a relativistic version of the Cowling approximation, in which all the metric perturbations are omitted in the pulsation equations (see McDermott, Van Horn, & Scholl 1983, and Finn 1988). The relativistic Cowling approximation is accurate enough for the f - and p -modes in non-rotating stars (Lindblom & Splinter 1992). It is also the case for the modes in slowly rotating stars (Yoshida & Kojima 1997). The relativistic Cowling approximation is a good approximation for oscillation modes in which the fluid motions are dominating over the metric fluctuations to determine the oscillation frequency. Therefore, it is justified to employ the relativistic Cowling approximation for the r -modes, for which the fluid motion is dominating.

If we employ the Cowling approximation, we can obtain our basic equations for pulsations from the perturbed energy and momentum conservation laws:

$$\delta(u^\alpha \nabla_\beta T_\alpha^\beta) = 0, \quad (\text{energy conservation law}) \quad (4)$$

$$\delta(q_\gamma^\alpha \nabla_\beta T_\alpha^\beta) = 0, \quad (\text{momentum conservation law}) \quad (5)$$

where ∇_α is the covariant derivative associated with the metric, T_β^α is the energy-momentum tensor, and q_β^α is the projection tensor with respect to the fluid four-velocity. Here, δQ denotes the Eulerian change in the physical quantity Q . In this paper, we adapt the adiabatic condition for the pulsation:

$$\Delta p = \frac{p\Gamma}{\rho + p} \Delta \rho, \quad (6)$$

where Γ is the adiabatic index defined as

$$\Gamma = \frac{\rho + p}{p} \left(\frac{\partial p}{\partial \rho} \right)_{ad}, \quad (7)$$

and ΔQ stands for the Lagrangian change in the physical quantity Q . The relation between the Lagrangian and the Eulerian changes is given by the equation:

$$\Delta Q = \delta Q + \mathcal{L}_\zeta Q, \quad (8)$$

where \mathcal{L}_ζ is the Lie derivative along the Lagrangian displacement vector ζ^α , which is defined by the relation:

$$\delta \hat{u}^\alpha = q_\beta^\alpha \delta u^\beta = q_\beta^\alpha (\mathcal{L}_u \zeta)^\beta. \quad (9)$$

Notice that we have $\delta\hat{u}^\alpha = \delta u^\alpha$ in the Cowling approximation. Because we are interested in pulsations of stationary rotating stars, we can assume that all the perturbed quantities have time and azimuthal dependence given by $e^{i\sigma t + im\varphi}$, where m is a constant integer and σ is a constant frequency measured by an inertial observer at the spatial infinity. Because of this assumption, the relation between the Lagrangian displacement ζ^α and the velocity perturbation $\delta\hat{u}^\alpha$ reduces to an algebraic relationship:

$$\delta\hat{u}^\alpha = i\gamma\bar{\sigma}\zeta^\alpha, \quad (10)$$

where $\bar{\sigma}$ is the frequency defined in the corotating frame defined as $\bar{\sigma} = \sigma + m\Omega$. Note that the gauge freedom in ζ^α has been used to demand the relation $u_\alpha\zeta^\alpha = 0$. By substituting equations (6) and (10) into equations (4) and (5), we can obtain the perturbed energy equation,

$$\frac{1}{\gamma}\nabla_\alpha(\gamma\zeta^\alpha) + \frac{1}{p\Gamma}(\delta p + \zeta^\alpha\nabla_\alpha p) = 0, \quad (11)$$

and the perturbed momentum equation,

$$-\gamma^2\bar{\sigma}^2 g_{\alpha\beta}\zeta^\beta + 2i\gamma\bar{\sigma}\zeta^\beta\nabla_\beta u_\alpha + q_\alpha^\beta\nabla_\beta\left(\frac{\delta p}{\rho+p}\right) + \left(\frac{\delta p}{\rho+p}q_\alpha^\beta + \zeta^\beta\frac{\nabla_\alpha p}{\rho+p}\right)A_\beta = 0, \quad (12)$$

where A_α is the relativistic Schwarzschild discriminant defined by

$$A_\alpha = \frac{1}{\rho+p}\nabla_\alpha\rho - \frac{1}{\Gamma p}\nabla_\alpha p. \quad (13)$$

Notice that equations (11) and (12) have been derived without the assumption of slow rotation. Physically acceptable solutions of equations (11) and (12) must satisfy boundary conditions at the center and the surface of the star. The surface boundary condition at $r = R$ is given by

$$\Delta p = \delta p + \zeta^\alpha\nabla_\alpha p = 0, \quad (14)$$

and the inner boundary condition is that all the eigenfunctions are regular at the center ($r = 0$).

On the analogy between general relativity and Newtonian gravity, the second term on the left hand side of equation (12) is interpreted as a relativistic counterpart of the Coriolis force. In our formulation, the terms due to the Coriolis like force are included in the perturbation equations, but the terms due to the centrifugal like force, which are proportional to $\Omega^2/(GM/R^3)$, are all ignored. In Newtonian theory of oscillations, this approximation is justified for low frequency modes satisfying the conditions $|2\Omega/\bar{\sigma}| \geq 1$ and $\Omega^2/(GM/R^3) \ll 1$

(Lee & Saio 1986; Unno et al. 1989; Bildsten, Ushomirsky & Cutler 1996; Yoshida & Lee 2001). In general relativity, we note that it is difficult to make a clear distinction between inertial forces such as the Coriolis force and the centrifugal force. From a physical point of view, however, it is also acceptable to use the approximation for low frequency oscillations in the background spacetime described by the metric (1).

The eigenfunctions are expanded in terms of spherical harmonic functions $Y_l^m(\theta, \varphi)$ with different values of l for a given m . The Lagrangian displacement, ζ^k and the pressure perturbation, $\delta p/(\rho + p)$ are expanded as

$$\zeta^r = r \sum_{l \geq |m|}^{\infty} S_l(r) Y_l^m(\theta, \varphi) e^{i\sigma t}, \quad (15)$$

$$\zeta^\theta = \sum_{l, l' \geq |m|}^{\infty} \left\{ H_l(r) \frac{\partial Y_l^m(\theta, \varphi)}{\partial \theta} - T_{l'}(r) \frac{1}{\sin \theta} \frac{\partial Y_{l'}^m(\theta, \varphi)}{\partial \varphi} \right\} e^{i\sigma t}, \quad (16)$$

$$\zeta^\varphi = \frac{1}{\sin^2 \theta} \sum_{l, l' \geq |m|}^{\infty} \left\{ H_l(r) \frac{\partial Y_l^m(\theta, \varphi)}{\partial \varphi} + T_{l'}(r) \sin \theta \frac{\partial Y_{l'}^m(\theta, \varphi)}{\partial \theta} \right\} e^{i\sigma t}, \quad (17)$$

$$\frac{\delta p}{\rho + p} = \sum_{l \geq |m|}^{\infty} \delta U_l(r) Y_l^m(\theta, \varphi) e^{i\sigma t}, \quad (18)$$

where $l = |m| + 2k$ and $l' = l + 1$ for even modes and $l = |m| + 2k + 1$ and $l' = l - 1$ for odd modes where $k = 0, 1, 2, \dots$ (Regge & Wheeler 1957; Thorne 1980). Here, even and odd modes are, respectively, characterized by their symmetry and antisymmetry of the eigenfunction with respect to the equatorial plane. Substituting the perturbed quantities (15)–(18) into linearized equations (11) and (12), we obtain an infinite system of coupled ordinary differential equations for the expanded coefficients. The details of our basic equations are given in the Appendix. Note that non-linear terms of $q \equiv 2\bar{\omega}/\bar{\sigma}$ where $\bar{\omega} \equiv \Omega - \omega$ are kept in our basic equations. For numerical calculations, the infinite set of ordinary differential equations are truncated to be a finite set by discarding all the expanding coefficients associated with l larger than l_{\max} , the value of which is determined so that the eigenfrequency and the eigenfunctions are well converged as l_{\max} increases (Yoshida & Lee 2000a).

3. r -Modes of Neutron Star Models

The neutron star models that we use in this paper are the same as those used in the modal analysis by McDermott, Van Horn, & Hansen (1988). The models are taken from the evolutionary sequences for cooling neutron stars calculated by Richardson et al. (1982),

where the envelope structure is constructed by following Gudmundsson, Pethick & Epstein (1983). These models are composed of a fluid core, a solid crust and a surface fluid ocean, and the interior temperature is finite and is not constant as a function of the radial distance r . The models are not barotropic and the Schwarzschild discriminant $|A|$ has finite values in the interior of the star. In order to avoid the complexity in the modal properties of relativistic r -modes brought about by the existence of the solid crust in the models (see Yoshida & Lee 2001), we treat the whole interior of the models as a fluid in the following modal analysis.

We computed frequency spectra of r -modes for the neutron star models called NS05T7, NS05T8, and NS13T8 (see, McDermott et al. 1988). The physical properties such as the total mass M , the radius R , the central density ρ_c , the central temperature T_c and the relativistic factor GM/c^2R are summarized in Table 1 (for other quantities, see McDermott et al. 1988). In Figures 1 and 2, scaled eigenfrequencies $\kappa \equiv \bar{\sigma}/\Omega$ of the r -modes of the three neutron star models are given as functions of $\hat{\Omega} \equiv \Omega/\sqrt{GM/R^3}$ for $m = 2$ and 3 cases, respectively. Here only the fundamental r -modes with $l' = m$ are considered because they are most important for the r -mode instability of neutron stars. We note that it is practically impossible to correctly calculate rotationally induced modes at $\hat{\Omega} \sim 0$ because of their coupling with high overtone g -modes having extremely low frequencies. From these figures, we can see that the scaled eigenfrequency κ is almost constant as $\hat{\Omega}$ varies. In other words, the relation $\bar{\sigma} \sim \kappa_0 \Omega$ is a good approximation for the fundamental r -modes with $l' = m$, where κ_0 is a constant. Comparing the two frequency curves, which nearly overlap each other, for the models NS05T7 and NS05T8, it is found that the detailed interior structure of the stars such as the temperature distribution $T(r)$ does not strongly affect the frequency of the fundamental r -modes with $l' = m$. This modal property is the same as that found for the fundamental $l' = m$ r -modes in Newtonian dynamics (see, Yoshida & Lee 2000b). On the other hand, comparing the frequency curves for the models NS05T7 (NS05T8) and NS13T8, we note that the r -mode frequency of relativistic stars is strongly dependent on the relativistic factor GM/c^2R of the models. This is because the values of the effective rotation frequency $\bar{\omega} \equiv \Omega - \omega$ in the interior is strongly influenced by the relativistic factor. Similar behavior of the GM/c^2R dependence of the r -mode frequency has been found in the analysis of Kojima's equation (Yoshida 2001; Ruoff & Kokkotas 2001a). In Table 2, the values of κ_0 for the r -modes shown in Figures 1 and 2 are tabulated, where κ_0 are evaluated at $\hat{\Omega} = 0.1$. The boundary values for the continuous part of the frequency spectrum for the $l' = m = 2$ r -modes derived from Kojima's equation (Kojima 1998) are also listed in the same table. As shown by Kojima (1998) (see, also, Beyer & Kokkotas 1999; Lockitch et al. 2001), if an r -mode falls in the frequency region bounded by the boundary values, Kojima's equation becomes singular and yields a continuous frequency spectrum and singular eigenfunctions as solutions. Although all the r -modes obtained in the present study are in the bounded

frequency region, our numerical procedure shows that the r -modes obtained here are isolated and discrete eigenmodes. In fact, no sign of continuous frequency spectrum appears in the present numerical analysis (for a sign of the appearance of a continuous frequency spectrum in a numerical analysis, see Schutz & Verdaguer 1983).

In Figures 3 and 4, the eigenfunctions iT_2 for the $l' = m = 2$ fundamental r -modes in the neutron star models NS05T8 and NS13T8 at $\hat{\Omega} = 0.1$ are shown. We can confirm from these figures that the eigenfunctions show no singular property, even though the frequencies are in the continuous part of the spectrum associated with Kojima’s equation. These figures also show that the fluid motion due to the r -modes is more confined near the stellar surface for NS13T8 than for NS05T8. This suggests that the eigenfunctions iT_m tend to be confined to the stellar surface as the relativistic factor of the star increases. The same property appears in regular r -mode solutions derived from Kojima’s equation (Yoshida 2001; Ruoff & Kokkotas 2001a; Yoshida & Futamase 2001). But, in the present case, the confinement of the eigenfunction for NS13T8 is not so strong, even though the model NS13T8 is highly relativistic in the sense that the relativistic factor is as large as $GM/c^2R = 0.249$.

4. Discussion

In the neutron star models analyzed in the last section, the buoyant force is produced by thermal stratification in the star. However, as suggested by Reisenegger & Goldreich (1992), the buoyant force in the core of neutron stars might become even stronger if the effect of the smooth change of the chemical composition of charged particles (protons and electrons) in the core is taken into account. We thus think it legitimate to examine the effects of the enhanced buoyant force on the modal properties of the relativistic r -modes. Although our neutron star models do not provide enough information regarding the composition gradient, we may employ for this experiment an approximation formula for the Schwarzschild discriminant due to the composition gradient in the core given by Reisenegger & Goldreich (1992):

$$rA_r = 3.0 \times 10^{-3} \left(\frac{\rho}{\rho_{nuc}} \right) \frac{r}{\rho} \frac{d\rho}{dr}, \quad (19)$$

where ρ_{nuc} denotes the nuclear density $\rho_{nuc} = 2.8 \times 10^{14} \text{ g cm}^{-3}$. In Figure 5, Brunt-Väisälä frequency due to the thermal stratification (solid line) and that due to the composition gradient (dashed line) are given as a function of $\log(1 - r/R)$ for model NS13T8, where the frequencies are normalized by $(GM/R^3)^{1/2}$. In this paper, the relativistic Brunt-Väisälä frequency N is defined as

$$N^2 = \frac{A_r}{\rho + p} \frac{dp}{dr}. \quad (20)$$

This figure shows that in the core the Brunt-Väisälä frequency due to the composition gradient is by several orders of magnitude larger and hence the corresponding buoyant force is much stronger than those produced by the thermal stratification. Note that the Brunt-Väisälä frequency due to the composition gradient becomes as large as $\sim (GM/R^3)^{1/2}$ in the core (see also, e.g., Lee 1995).

Replacing the Schwarzschild discriminant in the original neutron star models with that due to the composition gradient calculated by using equation (19), we computed the fundamental r -modes with $l' = m = 2$ for the models NS05T8 and NS13T8, and plotted $\kappa = \bar{\sigma}/\hat{\Omega}$ against rotation frequency $\hat{\Omega}$ in Figure 6, where κ 's for the original models with the thermal stratification were also plotted for convenience. As shown by the figure, κ slightly increases as $\hat{\Omega} \rightarrow 0$ for the models with the enhanced buoyant force. This is remarkable because such behavior of κ for the fundamental r -modes with $l' = m$ is not found in the case of Newtonian r -modes in non-barotropic stars (Yoshida & Lee 2000b). The slight increase of κ with decreasing $\hat{\Omega}$ may be explained in terms of the property of the eigenfunctions of the r -modes. We depict the eigenfunctions $i T_2$ of the $l' = m = 2$ fundamental r -modes in the neutron star model NS13T8 at $\hat{\Omega} = 0.2$ and .02 in Figures 7 and 8, where the solid line and dashed line in each figure denote the r -modes in the models with the compositional stratification and with the thermal stratification, respectively. As shown by Figure 7, in the case of rapid rotation, the buoyant force in the core does not strongly affect the properties of the eigenfunctions. On the other hand, in the case of slow rotation, Figure 8 shows that the amplitude of the eigenfunction is strongly confined to the region near the surface for the model with the enhanced buoyant force. For relativistic r -modes, as a result of the general relativistic frame dragging effect, the effective rotation frequency $\bar{\omega} = \Omega - \omega \propto \Omega$ acting on local fluid elements is an increasing function of the distance r from the stellar center, and thus fluid elements near the stellar surface are rotating faster than those near the stellar center. The oscillation frequency of the r -modes may be determined by the mean rotation frequency obtained by averaging local rotation frequencies over the whole interior of the star with a certain weighting function associated with the eigenfunction. In this case, it is reasonable to expect that the relativistic r -modes that have large amplitudes only in the regions near the surface get larger values of κ than those that have large amplitudes deep in the core.

For the models with the enhanced buoyant force, the current quadrupole moment of the r modes that determines the strength of the instability will be largely reduced when $\hat{\Omega}$ is small as a result of the strong confinement of the amplitudes. But, since the r -mode instability is believed to operate at rapid rotation rates, the effect of the amplitude confinement at small rotation rates may be irrelevant to the instability.

5. Conclusion

In this paper, we have investigated the properties of relativistic r -modes in slowly rotating neutron stars in the relativistic Cowling approximation by taking account of higher order effects of rotation than the first order of Ω . In our formalism, only the influence of the Coriolis like force on the oscillations are taken into account, and no effects of the centrifugal like force are considered. We obtain the fundamental r -modes associated with $l' = m = 2$ and 3 for three neutron star models. We find that the fundamental r -mode frequencies are in a good approximation given by $\bar{\sigma} \approx \kappa_0 \Omega$. The proportional coefficient κ_0 is only weakly dependent on Ω , but strongly depends on the relativistic parameter $GM/c^2 R$. For the fundamental r -modes with $l' = m$, we find that the buoyant force in the core is more influential to the relativistic r -modes than to the Newtonian r -modes. All the r -modes obtained in this paper are discrete modes with distinct regular eigenfunctions, and they all fall in the frequency range of the continuous spectrum of Kojima's equation. We may conclude that the fundamental r -modes obtained in this paper are the relativistic counterpart of the Newtonian r -modes.

Here, it is legitimate to mention the relation between the present work and the study by Kojima & Hosonuma (1999), who studied relativistic r -modes in slowly rotating stars in the Cowling approximation. By applying the Laplace transformation to linearized equations derived for the r -modes, Kojima & Hosonuma (1999) examined how a single component of initial perturbations with axial parity evolves as time goes, and showed that the perturbations cannot oscillate with a single frequency. In this paper, we have presented a formulation for small amplitude relativistic oscillations in rotating stars in the relativistic Cowling approximation, and solved the oscillation equations as a boundary-eigenvalue problem for the r -modes. In our formulation, we assume neither axially dominant eigenfunctions nor low frequencies to calculate the r -modes. Considering these differences in the treatment of the r -mode oscillations between the two studies, it is not surprising that our results are not necessarily consistent with those by Kojima & Hosonuma (1999).

Our results suggest that the appearance of singular r -mode solutions can be avoided by extending the original Kojima's equation so that terms due to the buoyant force in the stellar interior are included. As discussed recently by Yoshida & Futamase (2001) and Lockitch & Andersson (2001), we believe that the answer to the question whether r -mode oscillations in uniformly rotating relativistic stars show true singular behavior may be given by solving the fourth order ordinary differential equation derived by Kojima & Hosonuma (2000) for r -modes. Verification of this possibility remains as a future study.

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A. Basic Equations

We introduce column vectors \mathbf{y}_1 , \mathbf{y}_2 , \mathbf{h} , and \mathbf{t} , whose components are defined by

$$y_{1,k} = S_l(r), \quad (\text{A1})$$

$$y_{2,k} = \frac{1}{r \frac{d\nu}{dr}} \delta U_l(r), \quad (\text{A2})$$

$$h_{,k} = H_l(r), \quad (\text{A3})$$

and

$$t_{,k} = T_{l'}(r), \quad (\text{A4})$$

where $l = |m| + 2k - 2$ and $l' = l + 1$ for “even” modes, and $l = |m| + 2k - 1$ and $l' = l - 1$ for “odd” modes, and $k = 1, 2, 3, \dots$

In vector notation, equations for the adiabatic nonradial pulsation in a slowly rotating star are written as follows:

The perturbed energy equation (11) reduces to

$$r \frac{d\mathbf{y}_1}{dr} + \left(3 - \frac{V}{\Gamma} + r \frac{d\lambda}{dr} \right) \mathbf{y}_1 + \frac{V}{\Gamma} \mathbf{y}_2 - \mathbf{\Lambda}_0 \mathbf{h} + c_2 \hat{\omega} \hat{\sigma} (-m \mathbf{h} + \mathbf{C}_0 i \mathbf{t}) = 0. \quad (\text{A5})$$

The r component of the perturbed momentum equations (12) reduces to

$$r \frac{d\mathbf{y}_2}{dr} - (e^{2\lambda} c_1 \hat{\sigma}^2 + r A_r) \mathbf{y}_1 + (U + r A_r) \mathbf{y}_2 - c_1 \hat{\sigma}^2 \chi (-m \mathbf{h} + \mathbf{C}_0 i \mathbf{t}) = 0. \quad (\text{A6})$$

Here,

$$U = \frac{r \frac{d}{dr} (r \frac{d\nu}{dr})}{\frac{d\nu}{dr}}, \quad V = -\frac{d \ln p}{d \ln r}, \quad (\text{A7})$$

$$\chi = \frac{e^{2\nu}}{r} \frac{d}{dr} \left(r^2 e^{-2\nu} \frac{\bar{\omega}}{\bar{\sigma}} \right), \quad (\text{A8})$$

$$c_1 = \frac{r^2 e^{-2\nu}}{r \frac{d\nu}{dr}} \frac{M}{R^3}, \quad c_2 = r^2 e^{-2\nu} \frac{M}{R^3}, \quad (\text{A9})$$

and $\hat{\omega} \equiv \bar{\omega}/(GM/R^3)^{1/2}$, $\hat{\sigma} \equiv \bar{\sigma}/(GM/R^3)^{1/2}$, and $\hat{\sigma} \equiv \sigma/(GM/R^3)^{1/2}$ are frequencies in the unit of the Kepler frequency at the stellar surface, where $\bar{\omega} \equiv \Omega - \omega$ and $\bar{\sigma} \equiv \sigma + m\Omega$.

The θ and φ components of the perturbed momentum equations (12) reduce to

$$\mathbf{L}_0 \mathbf{h} + \mathbf{M}_1 i\mathbf{t} = \frac{1}{c_1 \hat{\sigma}^2} \mathbf{y}_2 + \{\mathbf{O} + \mathbf{M}_1 \mathbf{L}_1^{-1} \mathbf{K}\} \left(\chi \mathbf{y}_1 + \frac{c_2 \hat{\omega} \hat{\sigma}}{c_1 \hat{\sigma}^2} \mathbf{y}_2 \right), \quad (\text{A10})$$

$$\mathbf{L}_1 i\mathbf{t} + \mathbf{M}_0 \mathbf{h} = \mathbf{K} \left(\chi \mathbf{y}_1 + \frac{c_2 \hat{\omega} \hat{\sigma}}{c_1 \hat{\sigma}^2} \mathbf{y}_2 \right), \quad (\text{A11})$$

where

$$\mathbf{O} = m\mathbf{\Lambda}_0^{-1} - \mathbf{M}_1 \mathbf{L}_1^{-1} \mathbf{K}. \quad (\text{A12})$$

The quantities \mathbf{C}_0 , \mathbf{K} , \mathbf{L}_0 , \mathbf{L}_1 , $\mathbf{\Lambda}_0$, $\mathbf{\Lambda}_1$, \mathbf{M}_0 , \mathbf{M}_1 are matrices written as follows:

For even modes,

$$\begin{aligned} (\mathbf{C}_0)_{i,i} &= -(l+2)J_{l+1}^m, & (\mathbf{C}_0)_{i+1,i} &= (l+1)J_{l+2}^m, \\ (\mathbf{K})_{i,i} &= \frac{J_{l+1}^m}{l+1}, & (\mathbf{K})_{i,i+1} &= -\frac{J_{l+2}^m}{l+2}, \\ (\mathbf{L}_0)_{i,i} &= 1 - \frac{mq}{l(l+1)}, & (\mathbf{L}_1)_{i,i} &= 1 - \frac{mq}{(l+1)(l+2)}, \\ (\mathbf{\Lambda}_0)_{i,i} &= l(l+1), & (\mathbf{\Lambda}_1)_{i,i} &= (l+1)(l+2), \\ (\mathbf{M}_0)_{i,i} &= q\frac{l}{l+1}J_{l+1}^m, & (\mathbf{M}_0)_{i,i+1} &= q\frac{l+3}{l+2}J_{l+2}^m, \\ (\mathbf{M}_1)_{i,i} &= q\frac{l+2}{l+1}J_{l+1}^m, & (\mathbf{M}_1)_{i+1,i} &= q\frac{l+1}{l+2}J_{l+2}^m, \end{aligned}$$

where $l = |m| + 2i - 2$ for $i = 1, 2, 3, \dots$, and $q \equiv 2\bar{\omega}/\bar{\sigma}$, and

$$J_l^m \equiv \left[\frac{(l+m)(l-m)}{(2l-1)(2l+1)} \right]^{1/2}. \quad (\text{A13})$$

For odd modes,

$$\begin{aligned} (\mathbf{C}_0)_{i,i} &= (l-1)J_l^m, & (\mathbf{C}_0)_{i,i+1} &= -(l+2)J_{l+1}^m, \\ (\mathbf{K})_{i,i} &= -\frac{J_l^m}{l}, & (\mathbf{K})_{i+1,i} &= \frac{J_{l+1}^m}{l+1}, \\ (\mathbf{L}_0)_{i,i} &= 1 - \frac{mq}{l(l+1)}, & (\mathbf{L}_1)_{i,i} &= 1 - \frac{mq}{l(l-1)}, \\ (\mathbf{\Lambda}_0)_{i,i} &= l(l+1), & (\mathbf{\Lambda}_1)_{i,i} &= l(l-1), \\ (\mathbf{M}_0)_{i,i} &= q\frac{l+1}{l}J_l^m, & (\mathbf{M}_0)_{i+1,i} &= q\frac{l}{l+1}J_{l+1}^m, \end{aligned}$$

$$(\mathbf{M}_1)_{i,i} = q \frac{l-1}{l} J_l^m, \quad (\mathbf{M}_1)_{i,i+1} = q \frac{l+2}{l+1} J_{l+1}^m,$$

where $l = |m| + 2i - 1$ for $i = 1, 2, 3, \dots$

Eliminating \mathbf{h} and it from equations (A5) and (A6) by using equations (A10) and (A11), equations (A5) and (A6) reduce to a set of first-order linear ordinary differential equations for \mathbf{y}_1 and \mathbf{y}_2 as follows:

$$\begin{aligned} r \frac{d\mathbf{y}_1}{dr} &= \left\{ \left(\frac{V}{\Gamma} - 3 - r \frac{d\lambda}{dr} \right) \mathbf{1} + \chi (\mathcal{F}_{11} + c_2 \hat{\omega} \hat{\sigma} \mathcal{F}_{21}) \right\} \mathbf{y}_1 \\ &+ \left\{ \frac{1}{c_1 \hat{\sigma}^2} (\mathcal{F}_{12} + c_2 \hat{\omega} \hat{\sigma} \mathcal{F}_{22}) + \frac{c_2 \hat{\omega} \hat{\sigma}}{c_1 \hat{\sigma}^2} (\mathcal{F}_{11} + c_2 \hat{\omega} \hat{\sigma} \mathcal{F}_{21}) - \frac{V}{\Gamma} \mathbf{1} \right\} \mathbf{y}_2, \end{aligned} \quad (\text{A14})$$

$$\begin{aligned} r \frac{d\mathbf{y}_2}{dr} &= \left\{ (e^{2\lambda} c_1 \hat{\sigma}^2 + r A_r) \mathbf{1} - c_1 \hat{\sigma}^2 q^2 \mathcal{F}_{21} \right\} \mathbf{y}_1 \\ &+ \left\{ -(r A_r + U) \mathbf{1} - \chi (\mathcal{F}_{22} + c_2 \hat{\omega} \hat{\sigma} \mathcal{F}_{21}) \right\} \mathbf{y}_2, \end{aligned} \quad (\text{A15})$$

where

$$\mathcal{F}_{11} = \mathbf{W} \mathbf{O}, \quad (\text{A16})$$

$$\mathcal{F}_{12} = \mathbf{W}, \quad (\text{A17})$$

$$\mathcal{F}_{21} = \mathcal{R} \mathbf{W} \mathbf{O} - \mathbf{C}_0 \mathbf{L}_1^{-1} \mathbf{K}, \quad (\text{A18})$$

$$\mathcal{F}_{22} = \mathcal{R} \mathbf{W}, \quad (\text{A19})$$

$$\mathcal{R} = m \mathbf{\Lambda}_0^{-1} + \mathbf{C}_0 \mathbf{L}_1^{-1} \mathbf{M}_0 \mathbf{\Lambda}_0^{-1}, \quad (\text{A20})$$

$$\mathbf{W} = \mathbf{\Lambda}_0 (\mathbf{L}_0 - \mathbf{M}_1 \mathbf{L}_1^{-1} \mathbf{M}_0)^{-1}. \quad (\text{A21})$$

The surface boundary conditions $\Delta p(r = R) = 0$ are given by

$$\mathbf{y}_1 - \mathbf{y}_2 = 0. \quad (\text{A22})$$

The inner boundary conditions at the stellar center are the regularity conditions of the eigenfunctions.

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Table 1. Neutron Star Models

Model	M (M_\odot)	R (km)	ρ_c (g cm ³)	T_c (K)	$GM/(c^2 R)$
NS05T7	0.503	9.839	9.44×10^{14}	1.03×10^7	7.54×10^{-2}
NS05T8	0.503	9.785	9.44×10^{14}	9.76×10^7	7.59×10^{-2}
NS13T8	1.326	7.853	3.63×10^{15}	1.05×10^8	2.49×10^{-1}

Table 2. Scaled Eigenfrequencies κ_0 of Fundamental r -modes

Model	$\kappa_0(l = m = 2)$	$\kappa_0(l = m = 3)$	$2/3 \times \bar{\omega}(0)/\Omega$	$2/3 \times \bar{\omega}(R)/\Omega$
NS05T7	0.600	0.455	0.523	0.645
NS05T8	0.601	0.456	0.524	0.642
NS13T8	0.393	0.309	0.208	0.489

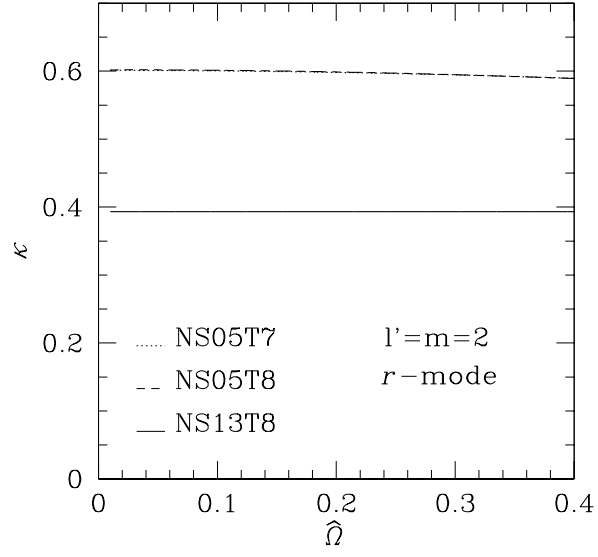


Fig. 1.— Scaled frequencies $\kappa = \bar{\sigma}/\Omega$ of the fundamental r -modes in the neutron star models NS05T7, NS05T8, and NS13T8 are plotted as functions of $\hat{\Omega} = \Omega/(GM/R^3)^{1/2}$ for $l' = m = 2$. Note that the two r -mode frequency curves for the models NS05T7 and NS05T8 overlap each other almost completely.

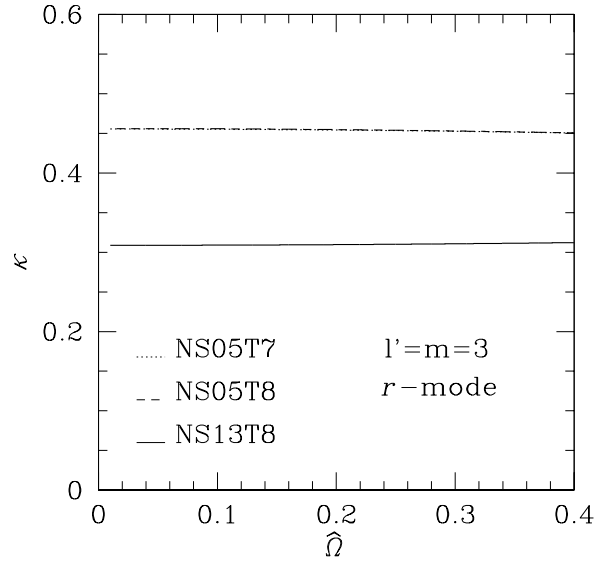


Fig. 2.— Same as Figure 1 but for the case of $m = 3$.

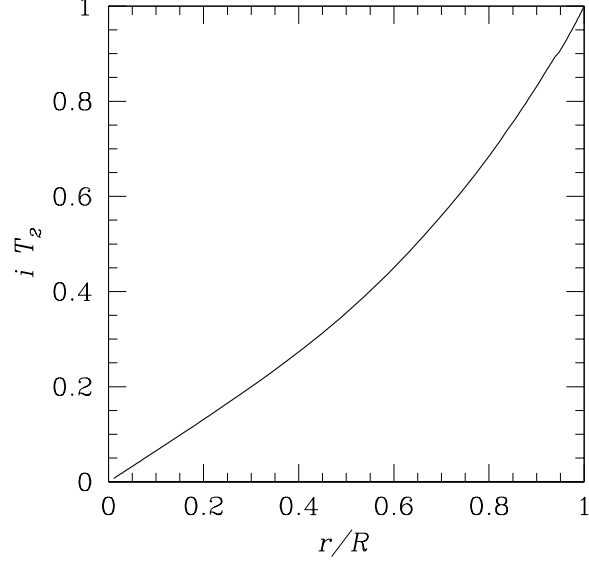


Fig. 3.— Eigenfunction $i T_2$ of the r -mode with $l' = m = 2$ for the model NS05T8 at $\hat{\Omega} = 0.1$ is given as a function of r/R . Here, normalization of the eigenfunction is chosen as $i T_2(R) = 1$.

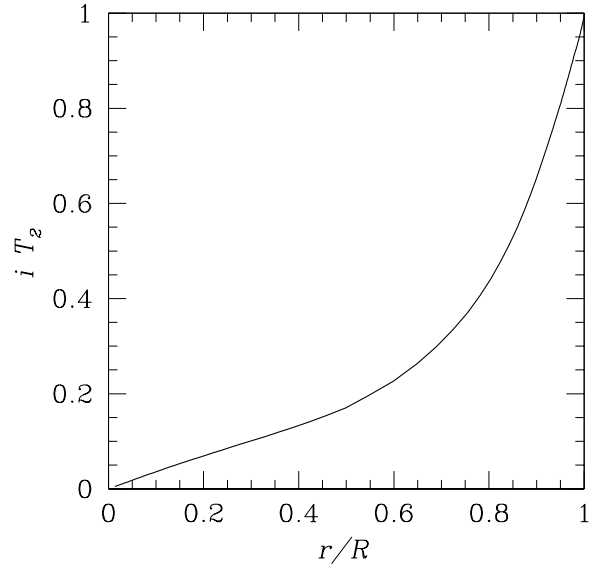


Fig. 4.— Same as Figure 4 but for the model NS13T8.

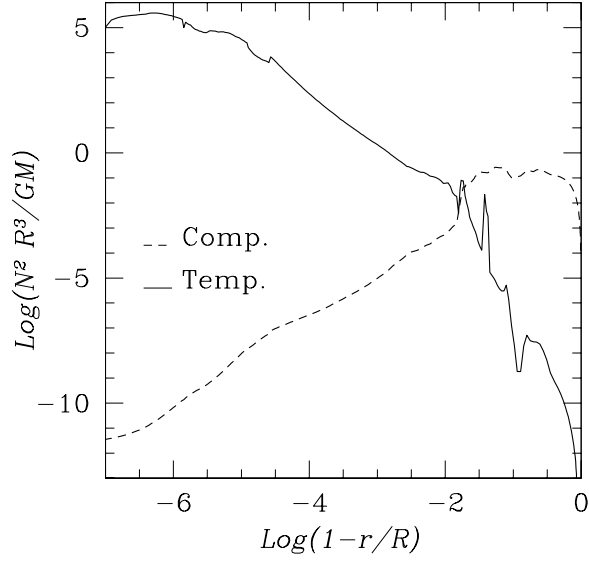


Fig. 5.— Brunt-Väisälä frequency due to the thermal stratification (solid line) and that due to the composition gradient (dashed line) are given as a function of $\log(1 - r/R)$ for model NS13T8, where the frequencies are normalized by $(GM/R^3)^{1/2}$.

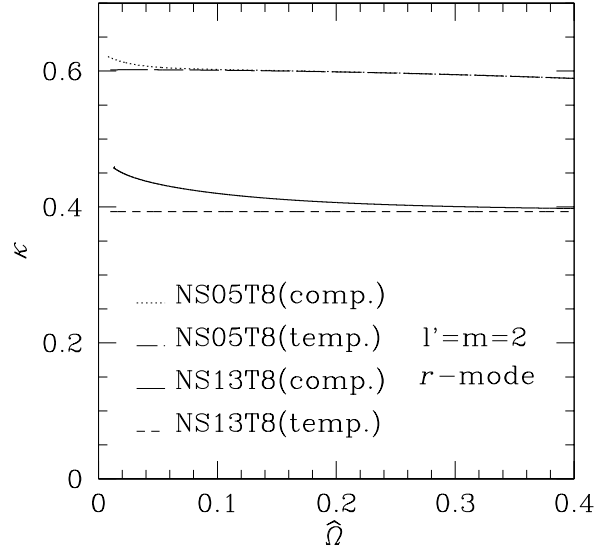


Fig. 6.— Scaled frequencies $\kappa = \bar{\sigma}/\Omega$ of the $l' = m = 2$ fundamental r -modes in the neutron star models NS05T8 and NS13T8 are plotted as functions of $\hat{\Omega} = \Omega/(GM/R^3)^{1/2}$. Labels ”comp.” and ”temp.” stand for the frequencies of the models with the buoyancy due to the composition gradient and due to the temperature gradient, respectively.

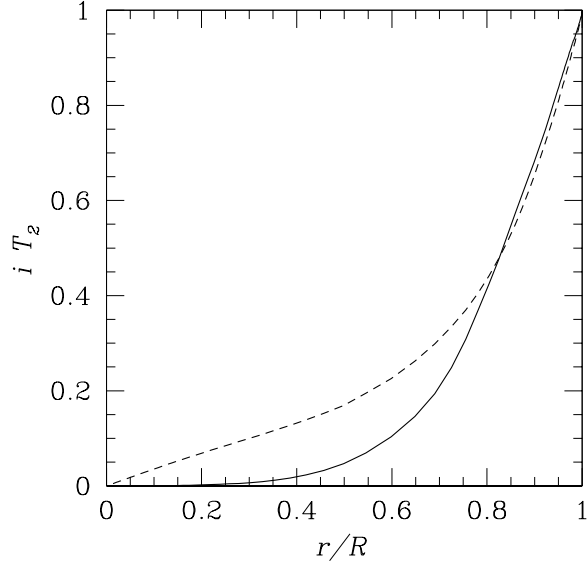


Fig. 7.— Eigenfunctions $i T_2$ of the $l' = m = 2$ fundamental r -modes in the neutron star model NS13T8 at $\hat{\Omega} = 0.2$ are given as a function of r/R . Here, normalization of the eigenfunction is chosen as $i T_2(R) = 1$. The solid line and dashed line denote the r -modes in the models with the compositional stratification and with the thermal stratification, respectively.

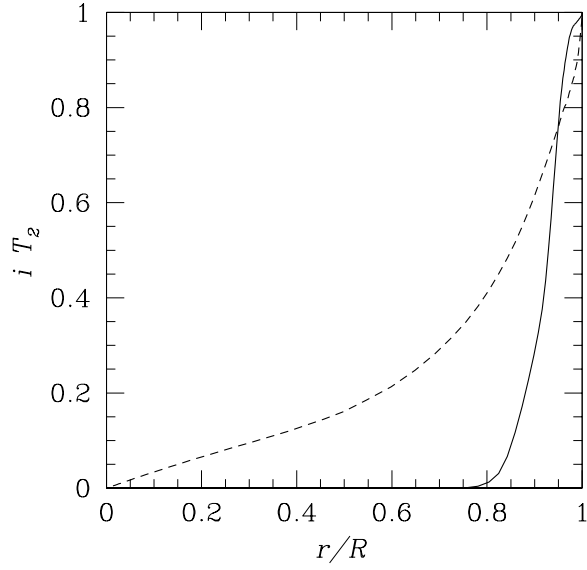


Fig. 8.— Same as Figure 7 but for the model at $\hat{\Omega} = 0.02$.